PAL
2/7/08

Unified Engineering Problem Set
Week 2 Spring 2008

SOLUTIONS

M2.1 (a) There are three key sets of equations:

• Equilibrium Equations $\left(\frac{\partial \sigma_{mn}}{\partial x_n} + f_m = 0\right)$

gives 3 equations

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + \frac{\partial \sigma_{13}}{\partial x_3} + f_1 = 0$$

$$\frac{\partial \sigma_{21}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \sigma_{23}}{\partial x_3} + f_2 = 0$$

$$\frac{\partial \sigma_{31}}{\partial x_1} + \frac{\partial \sigma_{32}}{\partial x_2} + \frac{\partial \sigma_{33}}{\partial x_3} + f_3 = 0$$

these are based on the fundamental of $\boxed{\text{equilibrium}}$

• Strain-Displacement $\left[\epsilon_{mn} = \frac{1}{2}\left(\frac{\partial u_m}{\partial x_n} + \frac{\partial u_n}{\partial x_m}\right)\right]$

gives 6 equations

$$\epsilon_{11} = \partial u_1 / \partial x_1$$

$$\epsilon_{22} = \partial u_2 / \partial x_2$$

$$\epsilon_{33} = \partial u_3 / \partial x_3$$

$$\epsilon_{21} = \epsilon_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right)$$

$$\epsilon_{31} = \epsilon_{13} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right)$$

$$\epsilon_{32} = \epsilon_{23} = \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right)$$

These are based on $\boxed{\text{geometrical relationships}}$ and have the key assumptions that strains are small such that angular changes are small. This can be measured/expressed as: $\cos\theta \approx 1$; $\sin\theta \approx \theta$

- Stress-Strain ($\sigma_{mn} = E_{mnpq}\epsilon_{pq}$)

gives 6 equations

$$\sigma_{11} = E_{1111}\epsilon_{11} + E_{1122}\epsilon_{22} + E_{1133}\epsilon_{33} + 2E_{1123}\epsilon_{23} + 2E_{1113}\epsilon_{13} + 2E_{1112}\epsilon_{12}$$

$$\sigma_{22} = E_{1122}\epsilon_{11} + E_{2222}\epsilon_{22} + E_{2233}\epsilon_{33} + 2E_{2223}\epsilon_{23} + 2E_{2213}\epsilon_{13} + 2E_{2212}\epsilon_{12}$$

$$\sigma_{33} = E_{1133}\epsilon_{11} + E_{2233}\epsilon_{22} + E_{3333}\epsilon_{33} + 2E_{3323}\epsilon_{23} + 2E_{3313}\epsilon_{13} + 2E_{3312}\epsilon_{12}$$

$$\sigma_{23} = E_{1123}\epsilon_{11} + E_{2223}\epsilon_{22} + E_{3323}\epsilon_{33} + 2E_{2323}\epsilon_{23} + 2E_{1323}\epsilon_{13} + 2E_{1223}\epsilon_{12}$$

$$\sigma_{13} = E_{1113}\epsilon_{11} + E_{2213}\epsilon_{22} + E_{3313}\epsilon_{33} + 2E_{2313}\epsilon_{23} + 2E_{1313}\epsilon_{13} + 2E_{1213}\epsilon_{12}$$

$$\sigma_{12} = E_{1112}\epsilon_{11} + E_{2212}\epsilon_{22} + E_{3312}\epsilon_{33} + 2E_{2312}\epsilon_{23} + 2E_{1312}\epsilon_{13} + 2E_{1212}\epsilon_{12}$$

These are based only on linear relationships between stress and strain (and are constitutive)

(b) Compatibility equations come from geometrical restrictions as manifested in the strain-displacement equations. Displacements must be continuous functions of the defined space (x_1, x_2, and x_3). Thus, with three such functions, the six strains cannot be independent as they derive from three independent functions. The compatibility equations relate the strain fields to be compatible with the continuity of the displacements.
There are 3 independent displacement functions and 6 strain-displacement relations. Thus, there must be 3 (= 6 − 3)

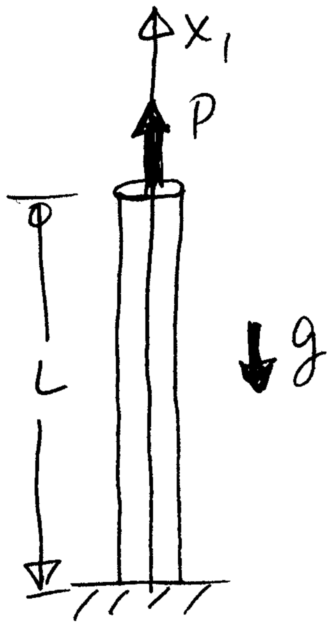
compatibility equations. They are derived by using the strain-displacement equations, taking "cross" derivatives and equating these.

These equation express *geometrical* restrictions.

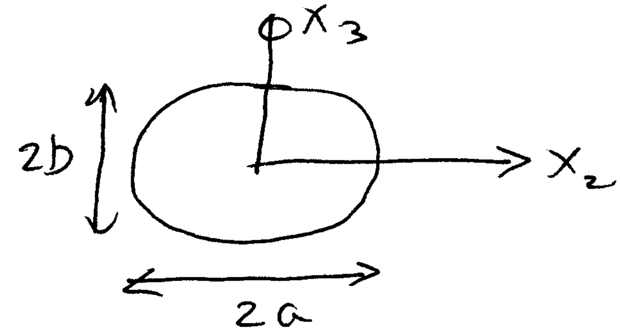
(c) In using engineering equations, the form of the equations change (e.g. σ_x rather than σ_{11}), but the underlying fundamentals and associated assumptions stay the same and the equations represent *the same thing*. Only the notation changes.

One key change due to definition is that *engineering shear* strain is $2 \times$ tensorial shear strain, so this factor of 2 must be incorporated in all equations with engineering shear strains.

M2.2



CROSS-SECTION



Aspect Ratio = 3/4

$$\Rightarrow \frac{2b}{2a} = \frac{3}{4} = 2b = \frac{3a}{2}$$

(a) Boundary Condition

@ $x_1 = 0$, rod is fixed to support

$\Rightarrow$ displacements are zero

$$\boxed{@\ x_1 = 0 : \quad u_1, u_2, u_3 = 0}$$

same for both cases

@ $x_1 = L$, condition changes for each case

$\rightarrow$ Tensile load P applied only:

force of magnitude P is applied over area $= \pi ab = \pi\left(\frac{2a}{2}\right)\left(\frac{3/2\,a}{2}\right) = \frac{3}{4}\pi a^2$

P is in x_1-direction

so:

$$\boxed{\begin{gathered}\sigma_{11} = \frac{P}{A} = \frac{4P}{3\pi a^2} \quad @\ x_1 = L \\ \sigma_{13}, \sigma_{12} = 0\end{gathered}}$$

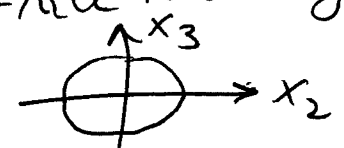
↳ no loads in other directions on x_1-surface

→ Mass of rod only:

@ $x_1 = L$, no loads are applied on x_1 surface. so:

$$\boxed{\sigma_{11} = \sigma_{12} = \sigma_{13} = 0 \quad @\ x_1 = L}$$

Along all other surfaces, all stresses on the surface are zero as no loads are applied. Boundary surface is along perimeter of ellipse



$$\boxed{\begin{gathered}\text{so: } @\ x_2, x_3 \text{ along ellipse perimeter} \\ \sigma_{22}, \sigma_{23}, \sigma_{33} = 0\end{gathered}}$$

same for both cases

(b) First consider the case of

→ Tensile load P applied only

⇒ no body forces.

Apply the equilibrium equations.
The only nonzero stress is σ_{11}, so:

$$\frac{\partial \sigma_{11}}{\partial x_1} + \underbrace{\cancel{f_1}}_{=0} = 0$$

integrating ⇒ σ_{11} = constant

Apply the B.C. @ $x_1 = L$: $\sigma_{11} = \frac{4P}{3\pi a^2}$

⇒ $\boxed{\sigma_{11} = \frac{4P}{3\pi a^2}}$

all other stresses are zero everywhere

To determine the strains, use the stress-strain relationships. For an isotropic material, need longitudinal (Young's) modulus, E, and Poisson's ratio, ν.

With σ_{11} the only nonzero stress:

$$\boxed{\epsilon_{12} = \epsilon_{13} = \epsilon_{23} = 0}$$ (no shear strains)

and:
$$\epsilon_{11} = \sigma_{11}/E$$
$$\epsilon_{22} = -\frac{\nu}{E}\sigma_{22}$$
$$\epsilon_{33} = -\frac{\nu}{E}\sigma_{33}$$

$$\Rightarrow \boxed{\begin{aligned} \epsilon_{11} &= \frac{4P}{3E\pi a^2} \\ \epsilon_{22} = \epsilon_{33} &= -\frac{4\nu P}{3E\pi a^2} \end{aligned}}$$

Now the...

→ Mass of rod only

We must now ~~consider~~ the body force (due to gravity):

$$f_1 = -\frac{\rho g \text{ volume}}{\text{volume}}$$ acts in negative x_1-direction

in the equilibrium equation. So:

$$\frac{\partial \sigma_{11}}{\partial x_1} - \rho g = 0$$

(all other terms are zero as the variation occurs along x_1)

Integrating this gives:

$$\sigma_{11} = \rho g x_1 + C_1$$

Again, apply the B.C. @ $x_1 = L$, $\sigma_{11} = 0$

$$\Rightarrow 0 = \rho g L + C$$

$$\Rightarrow C = -\rho g L$$

This gives:

$$\boxed{\sigma_{11} = \rho g (x_1 - L)}$$

(all other stresses are zero)

Notice: Value is negative for $x_1 < L \Rightarrow$ compressive stress. This makes physical sense.

The strains are again related through the stress-strain relationships. So we get:

$$\boxed{\begin{aligned} \epsilon_{11} &= \frac{\rho g}{E}(x_1 - L) \\ \epsilon_{22} = \epsilon_{33} &= -\frac{\nu \rho g}{E}(x_1 - L) \end{aligned}}$$

(again, $\epsilon_{12} = \epsilon_{13} = \epsilon_{23} = 0$)

Combining these for the

→ Full problem

requires only adding the solutions of the individual cases as all is linear and can be superposed. Thus:

$$\boxed{\sigma_{11} = \frac{4P}{3\pi a^2} + \rho g(x_1 - L)}$$

all other stresses are zero everywhere

$$\epsilon_{12} = \epsilon_{13} = \epsilon_{23} = 0$$

$$\epsilon_{11} = \frac{4P}{3E\pi a^2} + \frac{\rho g}{E}(x_1 - L)$$

$$\epsilon_{22} = \epsilon_{33} = \frac{-4\nu P}{3E\pi a^2} - \frac{\nu \rho g}{E}(x_1 - L)$$

Note: This can all be determined by combining the loading from the beginning and getting the combined boundary conditions and then applying the needed equations of elasticity to determine the stresses and strains.

(c) To find the displacements, apply the strain-displacement relations. The only primary consideration is u_1 so we use:

$$\epsilon_{11} = \frac{\partial u_1}{\partial x_1}$$

Again, consider the two loading cases separately

→ Tensile load P applied only

$$\epsilon_{11} = \frac{4P}{3E\pi a^2}$$

integration gives:

$$u_1 = \frac{4P}{3E\pi a^2} x_1 + C$$

constant of integration since no variation in x_2 and x_3

To find the constant, apply the B.C.:

@ $x_1 = 0$, $u_1 = 0$ $\Rightarrow$ $C = 0$

giving:

$$\boxed{u_1 = \frac{4P}{3E\pi a^2} x_1}$$

By definition of the model, u_2 and u_3 are 0. However, note the slight inconsistency since ϵ_{22} and ϵ_{33} are nonzero. Using:

$$\epsilon_{22} = \frac{\partial u_2}{\partial x_2}, \quad \epsilon_{33} = \frac{\partial u_3}{\partial x_3}$$

and integrating gives:

$$\epsilon_{22} = \frac{-4\nu P}{3E\pi a^2} x_2 + C_2$$

$$\epsilon_{33} = \frac{-4\nu P}{3E\pi a^2} x_3 + C_3$$

Define the midpoint of the area as the point of zero displacement (NOTE: one can define any point as reference point) to give $C_2 = 0$, $C_3 = 0$.

for the

→ Mass of rod only

the same equation and procedure applies. Here, the displacement becomes more complicated as ϵ_{11} is a function of x_1. So:

$$u_1 = \int \frac{\rho g}{E}(x_1 - L)\,dx_1$$

$$\Rightarrow u_1 = \frac{\rho g}{2E}x_1^2 - \frac{\rho g}{E}Lx_1 + C_1$$

Again, the B.C gives $u_1 = 0$ @ $x_1 = 0 \Rightarrow C_1 = 0$

so: $$\boxed{u_1 = \frac{\rho g}{E}x_1\left(\frac{x_1}{2} - L\right)}$$

Once again, the slight inconsistency dealing with u_2, u_3, ϵ_{22}, and ϵ_{33} exists. Expressions for u_2 and u_3 can be determined as in the previous case.

Now continuing for the

→ full problem

and considering the primary term, u_1, superposition can again be applied. So adding the results gives the overall:

$$\boxed{u_1 = \frac{4P}{3E\pi a^2} x_1 + \frac{\rho g}{E} x_1 \left(\frac{x_1}{2} - L\right)}$$

with the previously described inconsistencies concerning u_2 and u_3 existing in the same manner.

(d) There are always inconsistencies in the model with regard to the u_2 and u_3 displacements. This is built into the model. One must use the St. Venant's principle in the vicinity of the attachment to the support. "Away" from this region, the model is valid. "Near" the region, σ_{11} may vary with x_2 and x_3 and other stresses may be present. This will also be affected by the "part" of the problem being considered as the variability of the primary parameters in x_1 changes and this will relate to the importance of the inconsistencies (quantitatively). However, the way to address these do <u>not</u> change.

(e) This is simply an extension of the model to allow area to vary with x_1. In the general case, we replace the constant area we have ($A = \frac{3}{4}\pi a^2$) with a general functional relationship to x_1:

$$A = A(x_1)$$

Then we use this in the equation developed for the problem:

$$\sigma_{11} = \frac{P}{A(x_1)} - \int f_1(x_1)$$

It is noted that the body force also changes in x_1 due to the change in volume and area along x_1.

From the expression for stress, the strains are determined and subsequently the displacement.

In all this the difference is the variability of the area with x_1. An expression for that function, $A(x_1)$, is needed to be specific, but that is not necessary in order to explain the basic model applicability.

The inconsistencies involving the displacements u_2 and u_3 continue to exist. The inconsistencies may become more important or this will depend upon the rate that the area varies with x_1: $\partial A / \partial x_1$. Since area controls all the results.... it is in the denominator for all the key items. Finally, note that, if σ_{11} is a

function of x_1, then $\frac{\partial \sigma_{11}}{\partial x_1}$ is nonzero.
The first equilibrium equation is:

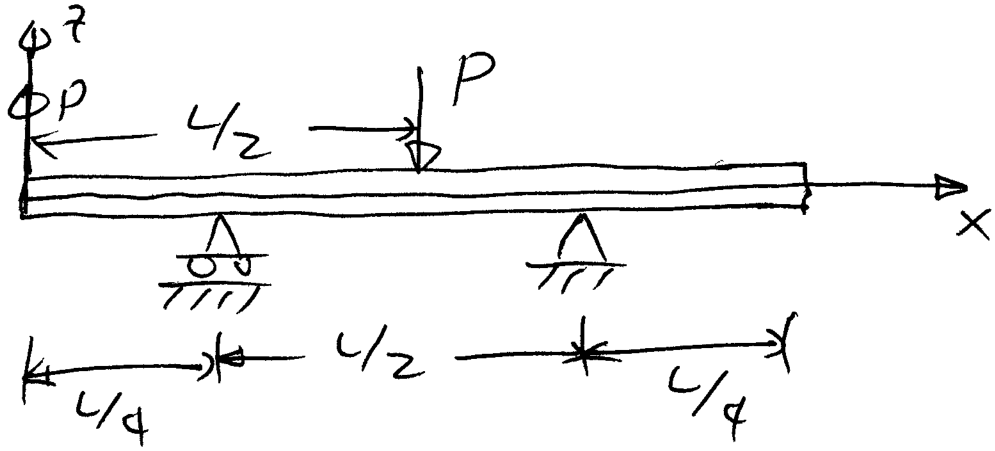
$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + \frac{\partial \sigma_{13}}{\partial x_3} = 0$$

and since the first term is now nonzero, one or both other terms must exist to satisfy equilibrium, so σ_{12} and/or σ_{13} exist! $\Rightarrow$ the model further falls apart ... it is not as applicable

$\rightarrow$ It comes down to the level of accuracy desired/needed and this can be initially assessed by looking at consistency.

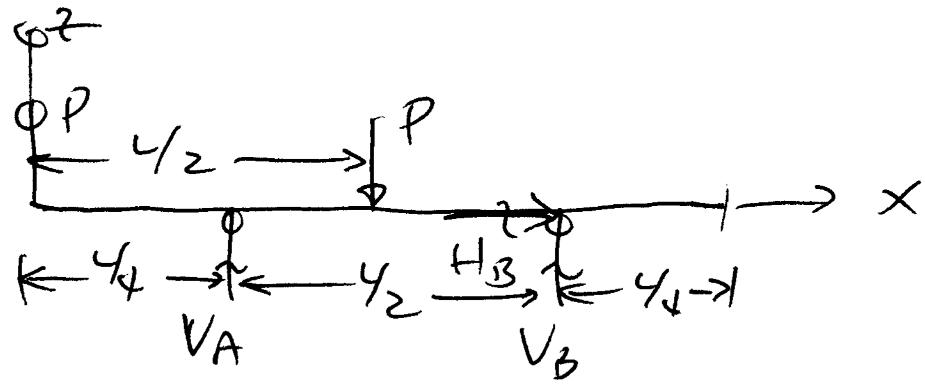
(f) To design the rod to be a structural fuse at the attachment to the ground, the maximum stress must occur at this point and it must be designed to reach σ_{ult} for the desired tensile load P_{fuse}. This is not just making a change in cross-sectional area that allows us to still apply our model (as discussed in section (e)) and making this the point of minimum area, but this must be done in such a way as to maximize σ_{11}. This is due to the fact that the stress due to the mass of the rod acts opposite (compressive) to the stress due to the applied load (tensile). Nevertheless applying the equations as developed herein and finding an area that maximizes stress at this point will yield the desired result.

M2.3



(a)

Draw the Free Body Diagram:



Take equilibrium:

$$\sum F_x = 0 \quad \xrightarrow{+} \quad \Rightarrow H_B = 0$$

$$\sum F_z = 0 \quad \uparrow + \quad \Rightarrow P + V_A - P + V_B = 0$$

$$\Rightarrow V_A + V_B = 0 \qquad (1)$$

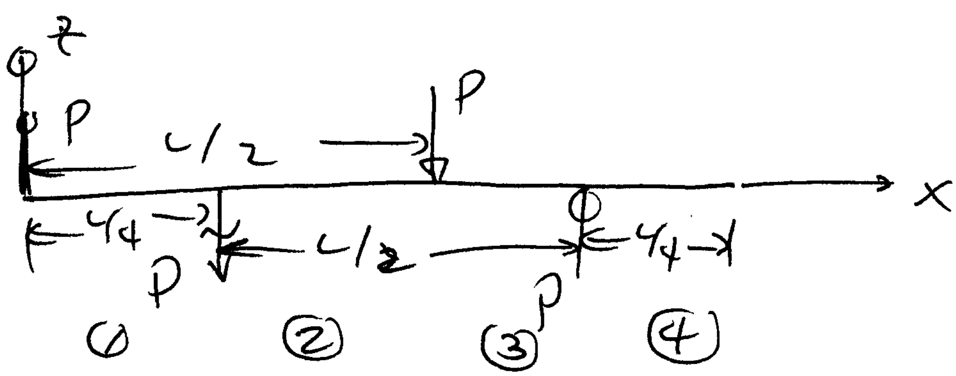
$$\Sigma M_A = 0 \circlearrowleft + \Rightarrow -PL/4 - PL/4 + V_B L/2 = 0$$

$$\Rightarrow V_B = P$$

apply equation (1): $\Rightarrow V_A = -P$

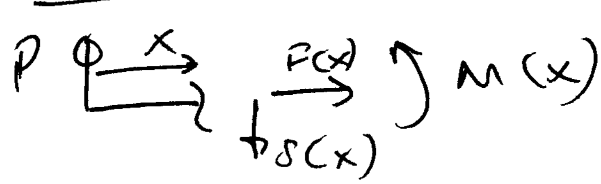
summarizing: $\boxed{\begin{aligned} V_A &= -P \\ V_B &= P \\ H_B &= 0 \end{aligned}}$

(b) First redraw the Free Body Diagram



Now take cuts in each of the four sections (as labeled) and apply equilibrium. The sections are defined by where loading changes.

→ <u>Section ①</u>: $0 < x < L/4$



$\sum F_x = 0 \overset{+}{\rightarrow} \Rightarrow F(x) = 0$

$\sum F_z = 0 \uparrow + \Rightarrow P - S(x) = 0$

$\Rightarrow S(x) = P$

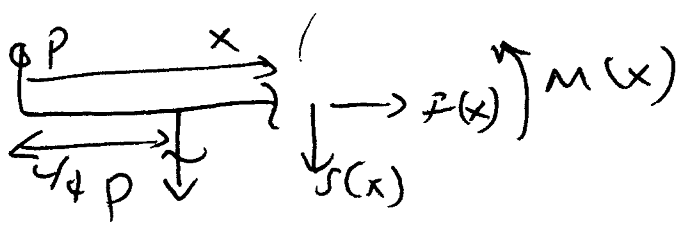
$\sum M_x = 0 \circlearrowleft + \Rightarrow -Px + M(x) = 0$

$\Rightarrow M(x) = Px$

So for $0 < x < L/4$ $\boxed{\begin{array}{l} S(x) = P \\ M(x) = Px \end{array}}$

with $F(x) = 0$

→ Section ②: $\frac{L}{4} < x < L/2$



$\sum F_x = 0 \overset{+}{\rightarrow} \Rightarrow F(x) = 0$

$\sum F_z : 0 \uparrow + \Rightarrow P - P + S(x) = 0 \Rightarrow S(x) = 0$

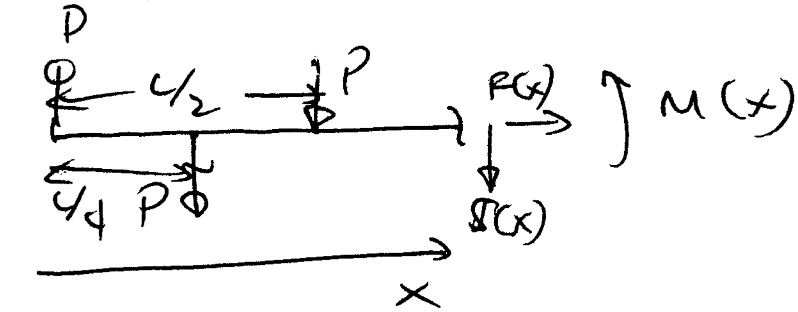
$\sum M_x = 0 \circlearrowleft + \Rightarrow -Px + P(x - L/4) + M(x) = 0$

$\Rightarrow M(x) = PL/4$

So for $L/4 < x < L/2$ $\boxed{\begin{array}{l} S(x) = 0 \\ M(x) = PL/4 \end{array}}$

with $F(x) = 0$

→ Section ③ : $\frac{L}{2} < x < \frac{3L}{4}$



$$\sum F_x = 0 \xrightarrow{+} \Rightarrow F(x) = 0$$

$$\sum F_z = 0 \uparrow + \Rightarrow P - P - P - S(x) = 0$$

$$\Rightarrow S(x) = -P$$

$$\sum M_x = 0 \circlearrowleft + \Rightarrow -Px + P(x - L/4) + P(x - L/2)$$

$$+ M(x) = 0$$

$$\Rightarrow Px - \frac{3}{4}PL + M(x) = 0$$

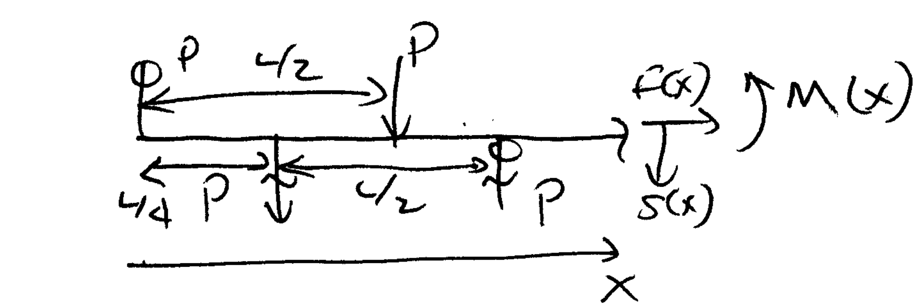
$$\Rightarrow M(x) = P\left(\frac{3}{4}L - x\right)$$

so for

$$\frac{L}{2} < x < \frac{3L}{4} \quad \boxed{\begin{aligned} S(x) &= -P \\ M(x) &= P\left(\tfrac{3}{4}L - x\right) \end{aligned}}$$

with $\cancel{F}F(x) = 0$

→ Section ④ : $\frac{3L}{4} < x < L$



$$\sum F_x = 0 \xrightarrow{+} \Rightarrow F(x) = 0$$

$$\sum F_z = 0 \uparrow + \Rightarrow P - P - P + P + S(x) = 0$$

$$\Rightarrow S(x) = 0$$

$$\sum M_x = 0 \circlearrowleft + \Rightarrow -Px + P(x - L/4) + P(x - L/2)$$

$$- P(x - 3L/4) + M(x) = 0$$

$$\Rightarrow M(x) = 0$$

so for

$$\frac{3L}{4} < x < L \quad \boxed{\begin{array}{l} S(x) = 0 \\ M(x) = 0 \end{array}}$$

with $F(x) = 0$

Note ways to check this:

1. At junctions of sections, the moments must be the same. They are in all cases.
2. At junctions of sections, the shears must change by any point load applied at that point. They do in all cases.
3. At the unloaded tip, the shear and moment must be zero. They are.

Finally, draw the diagrams: